Optimal bounds for geometric dilation and computer-assisted proofs

Discrete Mathematics Seminar - ULiège

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Triangulations

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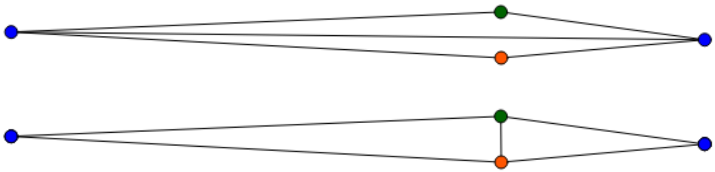

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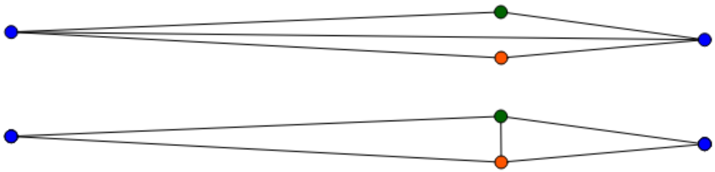

Definition

A **triangulation** of S is a planar network which is maximal for inclusion.

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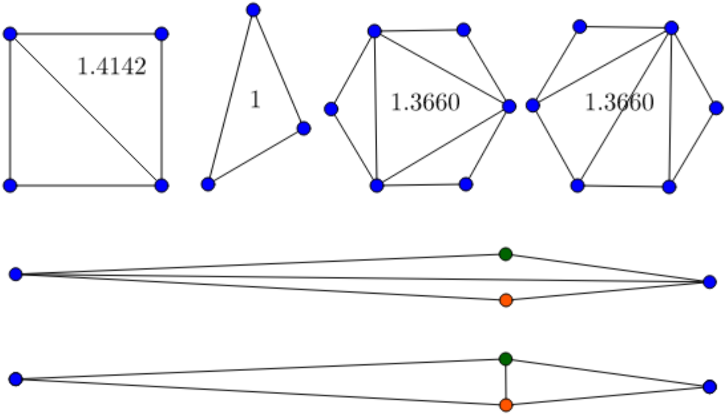
Goal

Find a triangulation T such that $\mathrm{dil}(T)$ is minimal:

$$\mathrm{dil}(S) := \min_{T \text{ triangulation of } S} \mathrm{dil}(T)$$

Examples



1 Geometric dilation of point sets

2 Degree-3 dilation of $\mathbb{Z}^2$

3 $\mathrm{dil}_3(\mathbb{Z}^2)$: dilation boost

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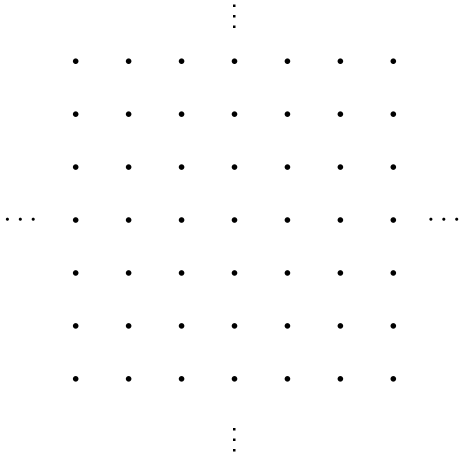
- **what about infinite point sets S?**

The square lattice: $S = \mathbb{Z}^2$



Previously known results about $\mathrm{dil}_k(\mathbb{Z}^2)$, $k \geq 4$

- Dumitrescu and Ghosh showed in [DG16a] that

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  requires to show the existence of triangulations with low dilation and degree $\leq k$, as was done in [DG16a].

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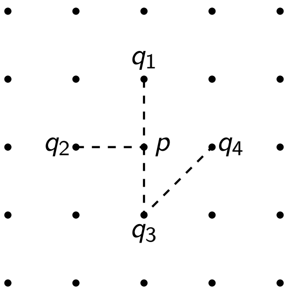

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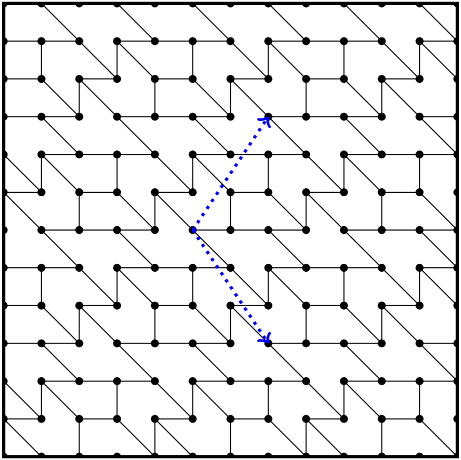
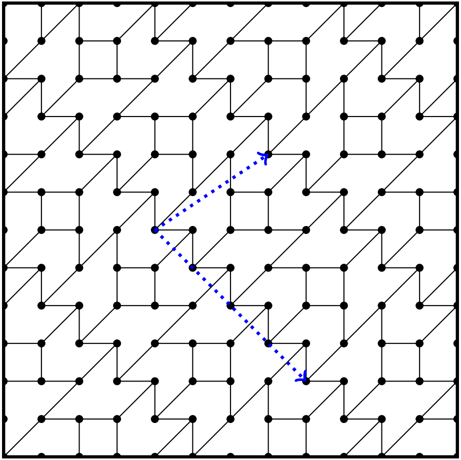
- With C. Pilatte, we *disproved* this conjecture by giving examples of degree-3 triangulations of $\mathbb{Z}^2$ with dilation $1+\sqrt{2}$.

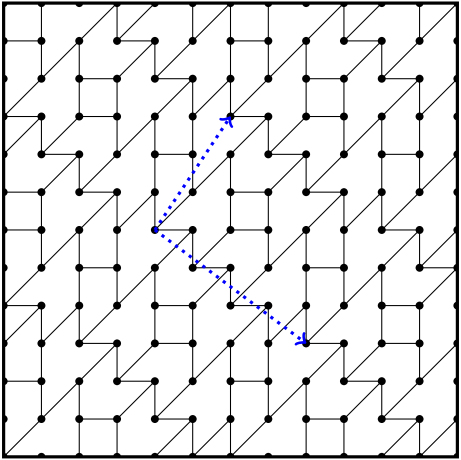
A periodic degree-3 triangulation of $\mathbb{Z}^2$ with dilation $1+\sqrt{2}$



Another example with dilation $1 + \sqrt{2}$



Yet another example



The computer-assisted search

Main ideas:

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- **Edges $\equiv$ obstructions to go from one side to the other;**
- Adding exhaustively “small tiles”, while respecting the degree 3 constraint, and try to detect pairs of points with high dilation as soon as possible (those with too many obstructions in between).
- The configurations are periodic, so we work on suitable “tori” with few points.

Optimal and locally optimal triangulations

Definition

Let $\mathcal{M}$ be the set of *optimal* triangulations, the triangulations on $\mathbb{Z}^2$ of maximum degree 3 which have dilation $1 + \sqrt{2}$, i.e. so that

$$d_T(p, q) \leq (1 + \sqrt{2})|pq|$$

for every pair of vertices $(p, q) \in \mathbb{Z}^2$.

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Definition

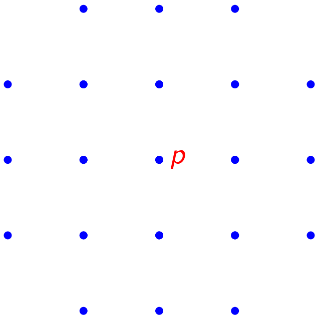
Let $\mathcal{M}_{\mathrm{loc}}$ be the set of *locally optimal* triangulations, the triangulations T on $\mathbb{Z}^2$ of maximum degree 3 which satisfy the dilation constraint

$$d_T(p,q) \leq (1+\sqrt{2})|pq|$$

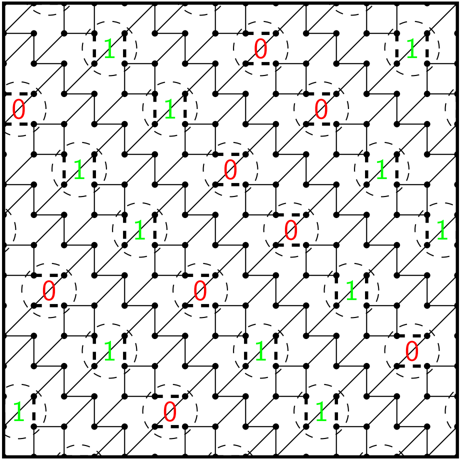
for every pair of vertices $(p,q) \in \mathbb{Z}^2$ with $|pq| \leq \sqrt{5}$.

Small zones considered in the definition of $\mathcal{M}_{\mathrm{loc}}$

Given $p \in \mathbb{Z}^2$, the blue dots represent the points $q \in \mathbb{Z}^2$ with $|pq| \leq \sqrt{5}$.



Uncountably many locally optimal triangulations



A structural result

Theorem ("Local-global principle"; G.-Pilatte 2022)

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Lemma ("Dilation boost")

Let $T \in \mathcal{M}_{\mathrm{loc}}$. *If* $p, q \in \mathbb{Z}^2$ *are such that* $|pq| = \sqrt{5}$, *then*

$$\frac{d_T(p,q)}{|pq|} \leq \frac{3+\sqrt{2}}{\sqrt{5}} \approx 1.974 < 2.414 \approx 1 + \sqrt{2}.$$

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Idea of the proof ot the Local-global principle.

If $p, q \in \mathbb{Z}^2$ are such that $|pq| > \sqrt{5}$, go from p to q using many "knight moves". Then $d_T(p,q)$ is small enough assuming the dilation boost. □

1 Geometric dilation of point sets

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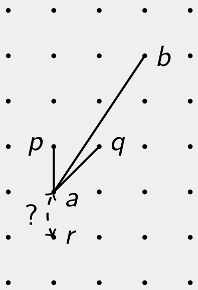
Some properties of triangulations in $\mathcal{M}_{\mathrm{loc}}$

Lemma

The edges of every $T \in \mathcal{M}_{\mathrm{loc}}$ are of length 1 or $\sqrt{2}$.

Proof.



Forbidden subconfigurations for triangulations of $\mathcal{M}_{\mathrm{loc}}$

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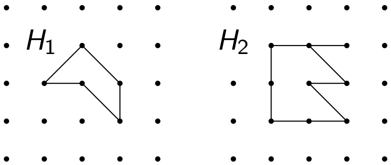
- Such forbidden subconfigurations will turn out to be crucial in the computer-assisted proof of the dilation boost.

Two forbidden subconfigurations

Lemma

Let $T \in \mathcal{M}_{\mathrm{loc}}$ and let H_1, H_2 be the following edge configurations. Then, neither H_1 nor H_2 (nor any translation, rotation or reflection of one of these two configurations) is a subgraph of T.



Proof.

Computer-assisted. □

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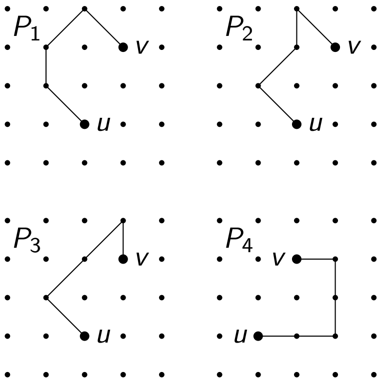
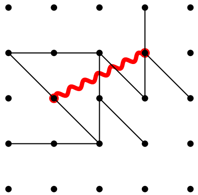
- This is not straightforward: a lengthy (luckily, computer-assisted!) exhaustive search needs to be performed to show that these configurations do not extend to any triangulation in $\mathcal{M}_{\mathrm{loc}}$;
- Without care, such an exhaustive search *does not terminate!* The tricky part is to choose well where to iterate over all possibilities to add an edge and to detect contradictions as soon as possible;

Computer-assisted proof of the dilation boost (1)

We fix two nodes u and v with $|uv| = \sqrt{5}$. The dilation boost says exactly that none of the following four paths can be a shortest path between u and v in a triangulation from $\mathcal{M}_{\mathrm{loc}}$.



Computer-assisted proof of the dilation boost (2)



Contradiction

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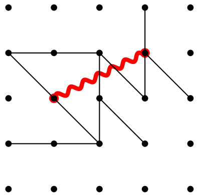

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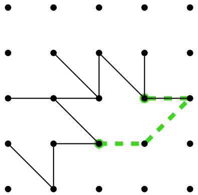

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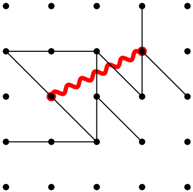

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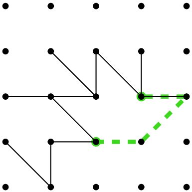

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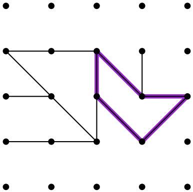

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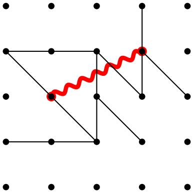

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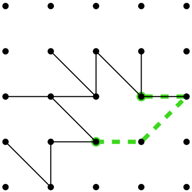

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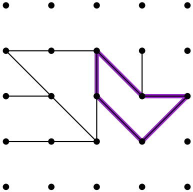

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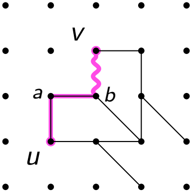

Shortcut

Computer-assisted proof of the dilation boost (3)

- We do an exhaustive search, but trying to detect contradictions as soon as possible, for instance *shortcuts* (when there is a too short path between u and v) or *contradictions* (when two points cannot be joined so that their dilation is $\leq 1 + \sqrt{2}$).

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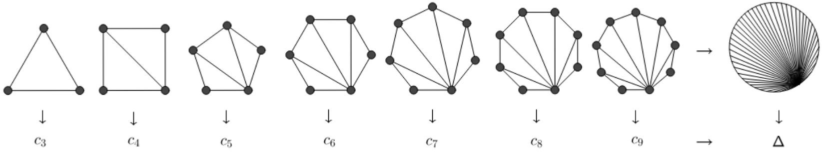
- The lemmas with the forbidden configurations are crucial: indeed, they "factorize" several impossible configurations that require quite a lot of computational work.
- Trying exhaustively to add edges in the right order is extremely important: not for correctness but for efficiency. If we do not go through the configuration in a "clever order", the search **never terminates**!

Thanks for your attention!

4 Dilation of a curve, the square

Dilation of regular polygons



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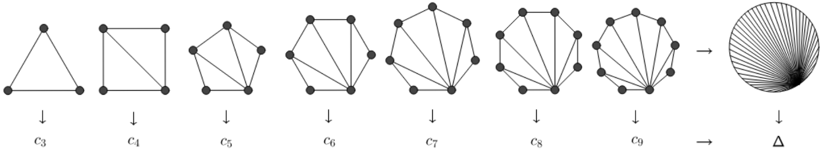

Theorem (2019; Pilatte)

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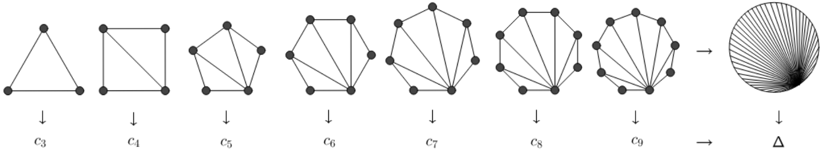

Theorem (2019; Pilatte)

The sequence of dilations of regular polygons converges to a value, ***the dilation of the circle.***

Dilation of the circle

- For each $n \geq 3$, we consider the dilation of the finite point set S_n whose vertices form a regular n-gon. We therefore consider a *sequence of combinatorial optimization problems*;

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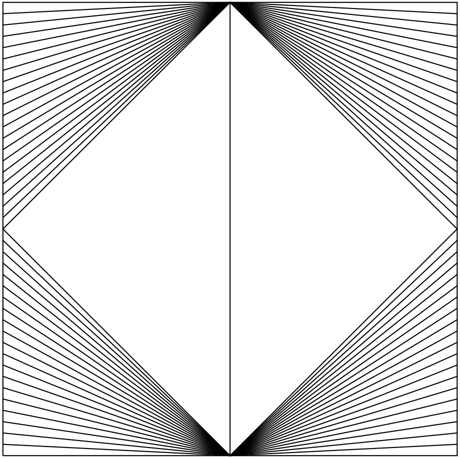
- There exists a *limit continuous optimization problem*, and there exists at least one optimal infinite triangulation (in a suitable precise sense) attaining the dilation of the circle;
- **Neither the dilation nor the optimal triangulation for the circle are known!**

Conjectured optimal triangulations for the square



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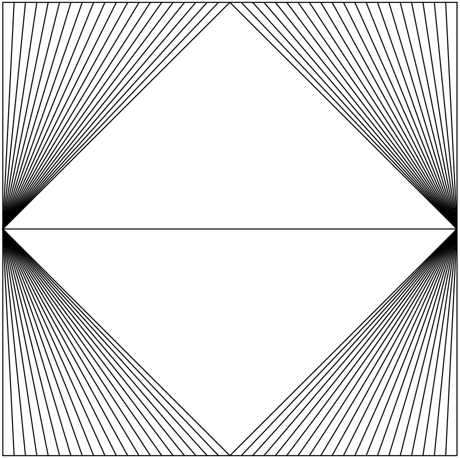
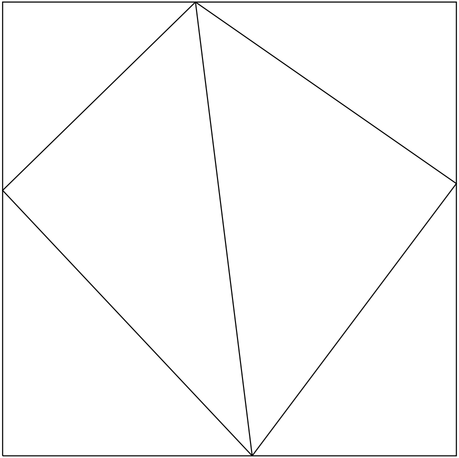


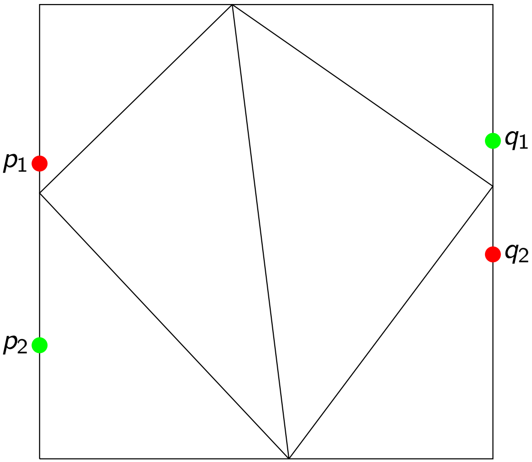
How to prove that those triangulations are optimal?

One can only consider triangulations containing a “central quadrilateral with a diagonal”:

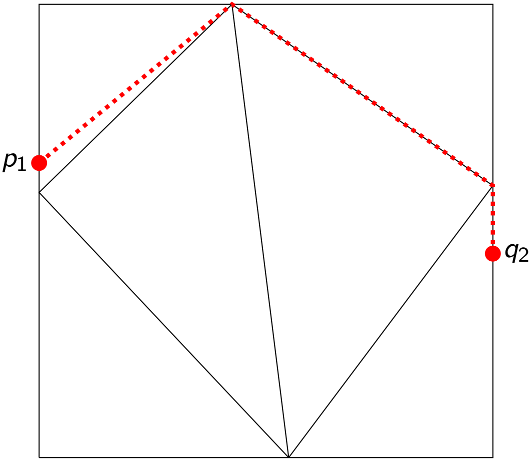


A pair of pairs

Two types of paths face a lot of obstruction: top-left to bottom-right and top-right to bottom-left:



Two paths for each pair



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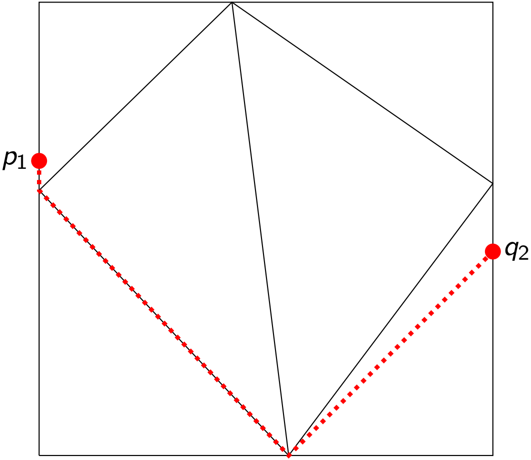

A “continuous” computer-assisted proof (work in progress)

- We need to show that the unique minimum of

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- A local analysis for (a,b,c,d) close $0_{\mathbb{R}^4}$ requires both theoretical and numerical ideas.

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